

MAXIMALLY AND NON-MAXIMALLY FAST ESCAPING POINTS OF TRANSCENDENTAL ENTIRE FUNCTIONS

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ABSTRACT. We partition the fast escaping set of a transcendental entire function into two subsets, the maximally fast escaping set and the non-maximally fast escaping set. These sets are shown to have strong dynamical properties. We show that the intersection of the Julia set with the non-maximally fast escaping set is never empty. The proof uses a new covering result for annuli, which is of wider interest.

It was shown by Rippon and Stallard that the fast escaping set has no bounded components. In contrast, by studying a function considered by Hardy, we give an example of a transcendental entire function for which the maximally and non-maximally fast escaping sets each have uncountably many singleton components.

1. INTRODUCTION

Suppose that $f : \mathbb{C} \rightarrow \mathbb{C}$ is a transcendental entire function. The *Fatou set* $F(f)$ is defined as the set $z \in \mathbb{C}$ such that $\{f^n\}_{n \in \mathbb{N}}$ is a normal family in a neighbourhood of z . The *Julia set* $J(f)$ is the complement in \mathbb{C} of $F(f)$. An introduction to the properties of these sets was given in [3].

For a general transcendental entire function the *escaping set*

$$I(f) = \{z : f^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty\}$$

was studied first in [7]. This paper concerns a subset of $I(f)$, the *fast escaping set* $A(f)$. This was introduced in [5], and can be defined [21] by

$$(1.1) \quad A(f) = \{z : \text{there exists } \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N}\}.$$

Here the *maximum modulus function* is defined by $M(r, f) = \max_{|z|=r} |f(z)|$, for $r \geq 0$. We write $M^n(r, f)$ to denote repeated iteration of $M(r, f)$ with respect to the variable r . In (1.1), $R > 0$ is such that $M^n(R, f) \rightarrow \infty$ as $n \rightarrow \infty$.

A major open question in transcendental dynamics is the conjecture of Eremenko [7] that, for every transcendental entire function f , $I(f)$ has no bounded components. A significant result regarding this conjecture was given by Rippon and Stallard, who showed [21, Theorem 1.1] that $A(f)$ has no bounded components. In view of this, the fast escaping set has been widely studied in recent years; see, for example, the papers [12, 16–21, 23] and [24].

We introduce a partition of $A(f)$, and show that the two sets in this partition share many properties with $A(f)$. On the other hand, we show that there is a transcendental entire function such that the components of these sets have unexpected boundedness properties.

2010 *Mathematics Subject Classification.* Primary 37F10; Secondary 30D05.

The author was supported by Engineering and Physical Sciences Research Council grant EP/J022160/1.

First we define

$$(1.2) \quad A'(f) = \{z \in A(f) : \exists N \in \mathbb{N} \text{ s.t. } |f^n(z)| = M(|f^{n-1}(z)|, f), \text{ for } n \geq N\},$$

and we let

$$A''(f) = A(f) \setminus A'(f).$$

We describe $A'(f)$ as the *maximally fast escaping set*, and $A''(f)$ as the *non-maximally fast escaping set*. A set S is *completely invariant* if $z \in S$ implies that $f(z) \in S$ and $f^{-1}(\{z\}) \subset S$. It is clear from the definitions that both $A'(f)$ and $A''(f)$ are completely invariant.

Our first result shows that, in some sense, $A'(f)$ is at most a small set.

Theorem 1. *If f is a transcendental entire function, then $A'(f)$ is contained in a countable union of curves each of which is analytic except possibly at its endpoints.*

In Example 1 we give a transcendental entire function such that $A'(f)$ has a single unbounded component, consisting of a countable union of analytic curves. It follows from Theorem 1 that this example is, in this sense, maximal.

It was shown in [5] that it follows from the construction in [7] that $A(f) \neq \emptyset$. In [17, Remark 2] it was further shown that $A(f) \cap J(f) \neq \emptyset$. In Example 2 we give a transcendental entire function such that $A'(f) = \emptyset$. On the other hand, using a new annuli covering result and a recent result regarding the properties of the boundary of a multiply connected Fatou component, we show that $A''(f) \cap J(f) \neq \emptyset$; roughly speaking, this means that there are always points in the Julia set for which the rate of escape is “fast” but not “maximally fast”.

Theorem 2. *If f is a transcendental entire function, then $A''(f) \cap J(f) \neq \emptyset$.*

It is known [21, Theorem 5.1(c)] that $A(f)$ is dense in $J(f)$, and [21, Theorem 1.2] that if U is a Fatou component that meets $A(f)$, then U is contained in $A(f)$. In the following theorem we strengthen these facts, and show that the sets $A'(f)$ and $A''(f)$ have strong dynamical properties relating to the Fatou and Julia sets. Note [21, Theorem 4.4] that all multiply connected Fatou components are in $A(f)$, and [4, 25] there exist transcendental entire functions with simply connected Fatou components contained in $A(f)$.

Theorem 3. *Suppose that f is a transcendental entire function. Then the following all hold.*

- (a) *If U is a simply connected Fatou component of f such that $U \cap A(f) \neq \emptyset$, then $U \subset A''(f)$.*
- (b) *If U is a multiply connected Fatou component of f , then $U \cap A''(f) \neq \emptyset$.*
- (c) *$A''(f)$ is dense in $J(f)$ and $J(f) \subset \partial A''(f)$.*
- (d) *If $A'(f) \cap J(f) \neq \emptyset$, then $A'(f)$ is dense in $J(f)$ and $J(f) \subset \partial A'(f)$.*
- (e) *If $A'(f) \cap F(f) = \emptyset$, then $J(f) = \partial A''(f)$.*
- (f) *If $A'(f) \cap J(f) \neq \emptyset$ and $A'(f) \cap F(f) = \emptyset$, then $J(f) = \partial A'(f)$.*

Clearly it follows from part (a) that if f has no multiply connected Fatou component, then $A'(f) \cap F(f) = \emptyset$. The additional hypothesis that $A'(f) \cap F(f) = \emptyset$ in parts (e) and (f) of Theorem 3 is required; in Example 3 we give a transcendental entire function, f , such that

$$A'(f) \cap F(f) \neq \emptyset, \quad J(f) \neq \partial A''(f) \text{ and } J(f) \neq \partial A'(f).$$

In Example 4 we give a transcendental entire function, f , which has a multiply connected Fatou component and which satisfies $A'(f) = \emptyset$.

We recall the property, mentioned earlier, that $A(f)$ has no bounded components. By studying a function considered by Hardy we show that this property does not hold for $A'(f)$ or $A''(f)$.

Theorem 4. *There is a transcendental entire function f such that*

- (a) $A'(f)$ is uncountable and totally disconnected;
- (b) $A''(f)$ has uncountably many singleton components and at least one unbounded component.

On the other hand, in Example 6 we give a transcendental entire function such that $A(f)$ has an unbounded component which is contained in $A'(f)$.

The structure of this paper is as follows. To prove Theorem 2 we require a new covering result for annuli, which is of independent interest. This is given in Section 2. In Section 3 we prove Theorem 1 and Theorem 2. In Section 4 we prove Theorem 3. Finally, in Section 5 we give the examples; the proof of Theorem 4 is included in Example 5.

2. A NEW COVERING RESULT

In this section we give a new covering result for annuli, which is used to construct a point in $A''(f) \cap J(f)$ in the case that f is a transcendental entire function with no multiply connected Fatou component. This result is similar to [15, Theorem 2.2], generalised to the whole family of iterates of a transcendental entire function. Here the *minimum modulus function* is defined, for $r \geq 0$, by $m(r, f) = \min_{|z|=r} |f(z)|$. We also use the notation for an annulus

$$A(r_1, r_2) = \{z : r_1 < |z| < r_2\}, \quad \text{for } 0 < r_1 < r_2,$$

and a disc

$$B(\zeta, r) = \{z : |z - \zeta| < r\}, \quad \text{for } 0 < r, \zeta \in \mathbb{C}.$$

Theorem 5. *Suppose that f is a transcendental entire function, and that*

$$1 < \lambda < \lambda' < \lambda''.$$

Then there exist $R' > 0$ and a function $\epsilon : \mathbb{R} \rightarrow \mathbb{R}$, both of which depend on f, λ, λ' and λ'' , such that the following holds. If $r \geq R'$, $n \in \mathbb{N}$, $\eta \geq 0$ and

$$(2.1) \quad \text{there exists } s \in (\lambda r, \lambda' r) \text{ such that } m(s, f^n) \leq \eta,$$

then there exists $w \in B(0, M^n(r, f))$ such that

$$f^n(A(r, \lambda'' r)) \supset B(0, M^n(r, f)) \setminus B(w, \epsilon(r) \max\{|w|, \eta\}).$$

Moreover, $\epsilon(r) \rightarrow 0$ as $r \rightarrow \infty$.

To prove Theorem 5 we require some results regarding the maximum modulus of a transcendental entire function. The first two are well-known:

$$(2.2) \quad \frac{\log M(r, f)}{\log r} \rightarrow \infty \text{ as } r \rightarrow \infty,$$

and

$$(2.3) \quad \frac{M(cr, f)}{M(r, f)} \rightarrow \infty \text{ as } r \rightarrow \infty, \quad \text{for } c > 1.$$

We require the following [19, Lemma 2.2].

Lemma 2.1. *If f is a transcendental entire function, then there exists $R_0 = R_0(f) > 0$ such that, for all $c > 1$,*

$$M(r^c, f) \geq M(r, f)^c, \quad \text{for } r \geq R_0.$$

We deduce the following by (2.2) and repeated application of Lemma 2.1.

Corollary 2.1. *If f is a transcendental entire function, then there exists $R_1 = R_1(f) > 0$ such that, for all $c > 1$ and all $n \in \mathbb{N}$,*

$$M^n(r^c, f) \geq M^n(r, f)^c, \quad \text{for } r \geq R_1.$$

Next, for a transcendental entire function f and for $c > 1$, we define a function

$$\psi_c(r) = \frac{1}{2} \left(\inf_{n \in \mathbb{N}} \frac{M^n(cr, f)}{M^n(r, f)} - 1 \right), \quad \text{for } r > 0.$$

We need the following lemma regarding the function ψ_c .

Lemma 2.2. *If f is a transcendental entire function and $c > 1$, then*

$$\psi_c(r) \rightarrow \infty \text{ as } r \rightarrow \infty.$$

Proof. By Corollary 2.1 and by (2.2), there exists $R = R(f) > 0$ such that, for all $n \in \mathbb{N}$ and $c > 1$,

$$\frac{M^n(cr, f)}{M^n(r, f)} \geq \frac{M^n(r, f)^{1+\log c / \log r}}{M^n(r, f)} = M^n(r, f)^{\log c / \log r} \geq M(r, f)^{\log c / \log r}, \quad \text{for } r \geq R.$$

By a second application of (2.2) we see that $M(r, f)^{\log c / \log r} \rightarrow \infty$ as $r \rightarrow \infty$. The result follows. \square

We also need the following, which is a version of part of [21, Lemma 2.6].

Lemma 2.3. *If f is a transcendental entire function and $d > 1$, then there exists $R_2 = R_2(f, d) > 0$ such that,*

$$M(dr, f^n) \geq M^n(r, f), \quad \text{for } r \geq R_2, \quad n \in \mathbb{N}.$$

Proof. The proof of this result is identical to the proof of the corresponding part of [21, Lemma 2.6], which gives the case that $d = 2$. \square

We also require some facts about the hyperbolic metric. If G is a hyperbolic domain and $z_1, z_2 \in G$, then we denote the density of the hyperbolic metric in G at z_1 by $\rho_G(z_1)$, and the hyperbolic distance between z_1 and z_2 in G by $[z_1, z_2]_G$. By [9, Theorem 9.13], there is an absolute constant $C > 1$ such that, with a suitable normalization of hyperbolic density, we have

$$(2.4) \quad \rho_{\mathbb{C} \setminus \{0,1\}}(z) \geq \frac{1}{2|z| \log(C|z|)}, \quad \text{for } z \in \mathbb{C} \setminus \{0,1\}.$$

For each $\tau > 1$, we define a constant $D_\tau > 1$ such that

$$(2.5) \quad \frac{1}{2} \log D_\tau = \max_{\{z, z' : |z|=|z'|=1\}} [z, z']_{A(1/\tau, \tau)}.$$

We now prove Theorem 5.

Proof of Theorem 5. Suppose that f is a transcendental entire function and that $1 < \lambda < \lambda' < \lambda''$. Set

$$\tau = \min \left\{ \lambda, \frac{\lambda''}{\lambda'} \right\} > 1 \quad \text{and} \quad c = \frac{1 + \lambda}{2} > 1.$$

By Lemma 2.2, we can choose R' sufficiently large that $\psi_c(r) > C^{D_\tau - 1}$, for $r \geq R'$. We also assume that $R' \geq R_2$, where $R_2 = R_2(f, d)$ is the constant from Lemma 2.3 with $d = \frac{\lambda}{c} > 1$.

Define the function $\epsilon : \mathbb{R} \rightarrow \mathbb{R}$ by

$$(2.6) \quad \epsilon(r) = \frac{2C^{1-1/D_\tau}}{\psi_c(r)^{1/D_\tau}}, \quad \text{for } r \geq R'.$$

Note that, since $C > 1$ and $D_\tau > 1$,

$$(2.7) \quad 2/\epsilon(r) < \psi_c(r), \quad \text{for } r \geq R'.$$

Note also, by Lemma 2.2, that $\epsilon(r) \rightarrow 0$ as $r \rightarrow \infty$.

Suppose that there exists $r \geq R'$, $n \in \mathbb{N}$, $\eta \geq 0$ and $s \in (\lambda r, \lambda' r)$ such that $m(s, f^n) \leq \eta$. We choose ζ and ζ' such that $|\zeta| = |\zeta'| = s$, and, by Lemma 2.3,

$$(2.8) \quad |f^n(\zeta)| \leq \eta \quad \text{and} \quad |f^n(\zeta')| = M(s, f^n) \geq M^n(cr, f).$$

Set $A = A(s/\tau, s\tau) \subset A(r, \lambda'' r)$. Suppose, by way of contradiction, that $f^n|_A$ omits values $w_1, w_2 \in B(0, M^n(r, f))$ such that

$$(2.9) \quad |w_2 - w_1| = \beta \max\{|w_1|, \eta\} \quad \text{where } \beta \geq \epsilon(r).$$

It follows by the contraction of the hyperbolic metric [6, Theorem 4.1] that

$$(2.10) \quad [\zeta, \zeta']_A \geq [f^n(\zeta), f^n(\zeta')]_{f^n(A)} > [\phi(f^n(\zeta)), \phi(f^n(\zeta'))]_{\mathbb{C} \setminus \{0, 1\}},$$

where $\phi(w) = (w - w_1)/(w_2 - w_1)$. By (2.8) and (2.9) we have

$$|\phi(f^n(\zeta))| \leq \frac{|f^n(\zeta)| + |w_1|}{|w_2 - w_1|} \leq \frac{\eta + |w_1|}{\beta \max\{|w_1|, \eta\}} \leq \frac{2}{\beta}$$

and

$$|\phi(f^n(\zeta'))| \geq \frac{|f^n(\zeta')| - |w_1|}{|w_2| + |w_1|} \geq \frac{M^n(cr, f) - M^n(r, f)}{2M^n(r, f)} \geq \psi_c(r).$$

Note that, by (2.7) and (2.9), we have that $\psi_c(r) > 2/\beta$. Suppose first that $2/\beta \geq 1$. Then, by (2.4), (2.5) and (2.10) we deduce that

$$\frac{1}{2} \log D_\tau > \int_{2/\beta}^{\psi_c(r)} \frac{dt}{2t \log(Ct)} = \frac{1}{2} \log \frac{\log(C\psi_c(r))}{\log(2C/\beta)},$$

and hence $\beta < \epsilon(r)$, which is a contradiction. On the other hand, if $2/\beta < 1$, then we deduce similarly that

$$\frac{1}{2} \log D_\tau > \int_1^{\psi_c(r)} \frac{dt}{2t \log(Ct)} = \frac{1}{2} \log \frac{\log(C\psi_c(r))}{\log C},$$

which is a contradiction to our choice of R' . \square

In our proof of Theorem 2, we use the following immediate corollary of Theorem 5. This is a generalisation of [15, Corollary 2.3], which considered the case $n = 1$. Here we use the following notation for a closed annulus

$$\overline{A}(r_1, r_2) = \{z : r_1 \leq |z| \leq r_2\}, \quad \text{for } 0 < r_1 < r_2.$$

Corollary 2.2. *Suppose that f is a transcendental entire function. Then there exists $R_3 = R_3(f) > 0$ such that the following holds. If there exists $r \geq R_3$, $n \in \mathbb{N}$ and $s \in (2r, 4r)$ such that $m(s, f^n) \leq 1$, and S, S', T, T' satisfy*

$$2 < S < S', \quad T < T' < M^n(r, f) \quad \text{and} \quad S' \leq \frac{1}{2}T,$$

then $f^n(A(r, 8r))$ contains $\overline{A}(S, S')$ or $\overline{A}(T, T')$.

3. PROPERTIES OF $A'(f)$ AND $A''(f)$

In this section we prove Theorem 1 and Theorem 2.

Proof of Theorem 1. Adapting the notation in [26], let $\mathcal{M}(f)$ be the set of points where a transcendental entire function, f , attains its maximum modulus; that is,

$$(3.1) \quad \mathcal{M}(f) = \{z : |f(z)| = M(|z|, f)\}.$$

It is well-known – see, for example, [27, Theorem 10] – that $\mathcal{M}(f)$ consists of, at most, a countable union of *maximal curves*, which are analytic except possibly at their endpoints.

If $z \in A'(f)$, where f is a transcendental entire function, then, by (1.2), there exists $N \in \mathbb{N}$ such that

$$(3.2) \quad f^n(z) \in \mathcal{M}(f), \quad \text{for } n \geq N.$$

In particular, it follows that,

$$A'(f) \subset \bigcup_{k=0}^{\infty} f^{-k}(\mathcal{M}(f)).$$

Hence $A'(f)$ is contained in a countable union of curves, which are analytic except possibly at their endpoints, as required. \square

In order to prove Theorem 2 we require a number of preliminary results, the first of which is a version of [20, Lemma 1], which considered images of sets under a single iteration of f .

Lemma 3.1. *Suppose that $(E_n)_{n \in \mathbb{N}}$ is a sequence of compact sets and $(m_n)_{n \in \mathbb{N}}$ is a sequence of integers. Suppose also that f is a transcendental entire function such that $E_{n+1} \subset f^{m_n}(E_n)$, for $n \in \mathbb{N}$. Set $p_n = \sum_{k=1}^n m_k$, for $n \in \mathbb{N}$. Then there exists $\zeta \in E_1$ such that*

$$(3.3) \quad f^{p_n}(\zeta) \in E_{n+1}, \quad \text{for } n \in \mathbb{N}.$$

If, in addition, $E_n \cap J(f) \neq \emptyset$, for $n \in \mathbb{N}$, then there exists $\zeta \in E_1 \cap J(f)$ such that (3.3) holds.

Proof. For $n \in \mathbb{N}$, we set

$$F_n = \{z \in E_1 : f^{p_1}(z) \in E_2, f^{p_2}(z) \in E_3, \dots, f^{p_n}(z) \in E_{n+1}\}.$$

It follows by hypothesis that $(F_n)_{n \in \mathbb{N}}$ is a decreasing sequence of non-empty compact subsets of E_1 , and so $F = \bigcap_{k=1}^{\infty} F_k$ is a non-empty subset of E_1 . We choose $\zeta \in F$ and the result follows.

Since $J(f)$ is completely invariant, the second statement follows by applying the first statement to the non-empty compact sets $E_n \cap J(f)$, for $n \in \mathbb{N}$. \square

We require some results concerning multiply connected Fatou components. The first is the following well-known result of Baker [2, Theorem 3.1]. We say that a set U surrounds a set V if V is contained in a bounded component of $\mathbb{C} \setminus U$. If U is a Fatou component, we write U_n , $n \geq 0$, for the Fatou component containing $f^n(U)$. We also let $\text{dist}(z, U) = \inf_{w \in U} |z - w|$, for $z \in \mathbb{C}$.

Lemma 3.2. *Suppose that f is a transcendental entire function and that U is a multiply connected Fatou component of f . Then each U_n is bounded and multiply connected, U_{n+1} surrounds U_n for large n , and $\text{dist}(0, U_n) \rightarrow \infty$ as $n \rightarrow \infty$.*

In addition we require [21, Theorem 4.4].

Lemma 3.3. *Suppose that f is a transcendental entire function, and that U is a multiply connected Fatou component of f . Then $\overline{U} \subset A(f)$.*

We also require some notation and results from [24]. Define a function R_A by
(3.4) $R_A(z) = \max\{R \geq 0 : M^n(R, f) \rightarrow \infty \text{ as } n \rightarrow \infty \text{ and } |f^n(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N}\},$
where we set $R_A(z) = -1$ if the set on the right-hand side of (3.4) is empty. If the set on the right-hand side of (3.4) is not empty, then the existence of a maximum follows from the continuity of $M(R, f)$.

If U is a multiply connected Fatou component which surrounds the origin, then we define $\partial_{\text{int}} U$ as the boundary of the component of $\mathbb{C} \setminus \overline{U}$ that contains the origin. We use the following lemma, which is a combination of [24, Lemma 4.2(c)] and [24, Lemma 4.4(a)].

Lemma 3.4. *Suppose that f is a transcendental entire function. Then there exists $R_4 = R_4(f) > 0$ such that the following holds. Suppose that U is a multiply connected Fatou component of f , which surrounds the origin and which satisfies $\text{dist}(0, U) \geq R_4$. Then there exists $R' \geq 0$ such that*

$$R_A(z) = R', \quad \text{for } z \in \partial_{\text{int}} U.$$

We also need the following simple result.

Lemma 3.5. *Suppose that f is a transcendental entire function, that $z \in A'(f)$ and that $R_A(z) \geq 0$. Then there exists $N \in \mathbb{N}$ such that*

$$|f^n(z)| = M^n(R_A(z), f), \quad \text{for } n \geq N.$$

Proof. Suppose that $z \in A'(f)$. It follows from (1.2) that there exists $N \in \mathbb{N}$ such that, with $R = |f^{N-1}(z)|$, we have $M^n(R, f) \rightarrow \infty$ as $n \rightarrow \infty$, and

$$|f^n(z)| = M^{n+1-N}(R, f), \quad \text{for } n \geq N.$$

Since $R_A(z) \geq 0$ we have, by (3.4), that

$$M^{N-1}(0, f) \leq M^{N-1}(R_A(z), f) \leq |f^{N-1}(z)| = R.$$

It follows, by the continuity of the function M , that there exists $R' \geq R_A(z)$ such that $M^{N-1}(R', f) = R$. We deduce that

$$|f^n(z)| = M^n(R', f), \quad \text{for } n \geq N.$$

The result follows since, by (3.4), we must have $R_A(z) = R'$. \square

We now prove Theorem 2.

Proof of Theorem 2. Let f be a transcendental entire function. Our proof splits into two cases. If f has a multiply connected Fatou component, then we show that there are, at most, countably many points in $A'(f) \cap J(f)$ on the inner boundary of some multiply connected Fatou component, which is a continuum in $A(f) \cap J(f)$. If f has no multiply connected Fatou component, then we use Theorem 5 to construct a point ζ which lies in $A''(f) \cap J(f)$.

Suppose first that f has a multiply connected Fatou component, U . By Lemma 3.2 we may assume that U surrounds the origin and that $\text{dist}(0, U) \geq R_4$, where R_4 is the constant from Lemma 3.4. It follows from Lemma 3.4 that there exists $R' \geq 0$ such that $R_A(z) = R'$, for $z \in \partial_{\text{int}} U$. We deduce by Lemma 3.5 that for each $z \in \partial_{\text{int}} U \cap A'(f)$ there exists $N = N(z) \in \mathbb{N}$ such that $|f^n(z)| = M^n(R', f)$, for $n \geq N$.

We recall the definition of the set $\mathcal{M}(f)$ in (3.1). It follows, by (3.2), that

$$(3.5) \quad \partial_{\text{int}} U \cap A'(f) \subset \bigcup_{k=0}^{\infty} f^{-k}(\{z \in \mathcal{M}(f) : |z| = M^k(R', f)\}).$$

It is known [27, Theorem 10] that, for each $R \geq 0$, the set $\{z \in \mathcal{M}(f) : |z| = R\}$ is finite. Hence the right-hand side of (3.5) is countable. Since, by Lemma 3.3, $\partial_{\text{int}} U$ is a continuum contained in $A(f) \cap J(f)$, we deduce that $\partial_{\text{int}} U \cap A''(f) \cap J(f)$ is uncountable.

On the other hand, suppose that f has no multiply connected Fatou component. Set $K = \frac{1}{2048}$ and let R_3 be the constant from Corollary 2.2. (We note that this choice of K is smaller than necessary for the proof of Theorem 2, but facilitates the proof of Lemma 5.2; see equations (3.11) and (5.16) below.) Define a function

$$\mu(r) = KM(r, f), \quad \text{for } r > 0.$$

We construct an increasing sequence of real numbers $(r_n)_{n \in \mathbb{N}}$ and a sequence of integers $(m_n)_{n \in \mathbb{N}}$. Choose $r_1 \geq R_3$ sufficiently large that $B(0, r_1) \cap J(f) \neq \emptyset$ and also, by (2.2), that

$$(3.6) \quad \mu(r) > \max\{r^2, 2\}, \quad \text{for } r \geq r_1.$$

By (2.3), we may also assume that r_1 is sufficiently large that

$$\mu(r) \geq \frac{1}{K}M(Kr, f), \quad \text{for } r \geq r_1.$$

We deduce that

$$(3.7) \quad \mu^n(r) \geq M^n(Kr, f), \quad \text{for } r \geq r_1, \quad n \in \mathbb{N}.$$

The construction proceeds inductively. Suppose that, for some $k \geq 1$, we have constructed the sequences $(r_n)_{n \leq k}$ and $(m_n)_{n < k}$. Let A be the annulus $A = A(r_k, 8r_k)$ and let A' be the annulus $A' = A(2r_k, 4r_k)$. Suppose that

$$m(s, f^n) > 1, \quad \text{for } s \in (2r_k, 4r_k), \quad n \in \mathbb{N}.$$

Then, by Montel's theorem, $\{f^n\}_{n \in \mathbb{N}}$ is a normal family in A' , and so $A' \subset F(f)$. Hence, by the choice of r_1 , A' is contained in a multiply connected Fatou component of f , which a contradiction. Therefore, there exists $m_k \in \mathbb{N}$ and $s \in (2r_k, 4r_k)$ such that $m(s, f^{m_k}) \leq 1$.

Set $S = KM^{m_k}(r_k, f)$, $S' = 8S$, $T = 16S$ and $T' = 128S$. It follows from Corollary 2.2 that $f^{m_k}(A(r_k, 8r_k))$ contains either $\bar{A}(S, S')$, in which case we set $r_{k+1} = S$, or $\bar{A}(T, T')$, in which case we set $r_{k+1} = T$. Note that in either case we have, by (3.6), that

$$r_{k+1} \geq KM^{m_k}(r_k, f) \geq \mu^{m_k}(r_k) > r_k.$$

This completes the construction of the sequences.

We now define a sequence of closed annuli $(E_n)_{n \in \mathbb{N}}$ by

$$E_n = \bar{A}(r_n, 8r_n), \quad \text{for } n \in \mathbb{N}.$$

It follows at once from the above construction that $E_{n+1} \subset f^{m_n}(E_n)$, for $n \in \mathbb{N}$. We also have, by the choice of r_1 and since f has no multiply connected Fatou component, that $E_n \cap J(f) \neq \emptyset$, for $n \in \mathbb{N}$. Hence, by Lemma 3.1, there exists $\zeta \in J(f)$ such that (3.3) holds with $p_n = \sum_{k=1}^n m_k$, for $n \in \mathbb{N}$.

We claim next that $\zeta \in A(f)$. We note, by (3.3) and by construction, that

$$\begin{aligned} |f^{p_n}(\zeta)| &\geq r_{n+1} \\ &\geq KM^{m_n}(r_n, f) \\ &\geq KM^{m_n}(KM^{m_{n-1}}(r_{n-1}, f), f) \\ &\vdots \\ &\geq \mu^{p_n}(r_1), \quad \text{for } n \in \mathbb{N}. \end{aligned}$$

By (3.6), there exists $\ell \in \mathbb{N}$ such that $\mu^\ell(r_1) \geq r_1/K$. We deduce by (3.7) that

$$(3.8) \quad \mu^{n+\ell}(r_1) \geq \mu^n\left(\frac{r_1}{K}\right) \geq M^n(r_1, f), \quad \text{for } n \in \mathbb{N}.$$

Hence

$$|f^{p_n}(\zeta)| \geq \mu^{p_n}(r_1) \geq M^{p_n-\ell}(r_1, f), \quad \text{for sufficiently large values of } n.$$

It follows that $|f^k(\zeta)| \geq M^{k-\ell}(r_1, f)$, for sufficiently large values of k , and so $\zeta \in A(f)$ as claimed.

Finally, suppose that $\zeta \in A'(f)$, in which case, by (1.2), there exists $N \in \mathbb{N}$ such that

$$(3.9) \quad |f^{n+p}(\zeta)| = M^n(|f^p(\zeta)|, f), \quad \text{for } n \in \mathbb{N} \text{ and } p \geq N.$$

However, by the choices of S , T and K , we have that $r_{n+1} \leq \frac{1}{128}M^{m_n}(r_n, f)$, for $n \in \mathbb{N}$, and so if $z \in E_n$ and $f^{m_n}(z) \in E_{n+1}$, for some $n \in \mathbb{N}$, then

$$(3.10) \quad |f^{m_n}(z)| \leq 8r_{n+1} \leq \frac{1}{16}M^{m_n}(r_n, f) \leq \frac{1}{16}M^{m_n}(|z|, f).$$

Hence, by (3.3) and (3.10) we have, for all sufficiently large $n \in \mathbb{N}$,

$$(3.11) \quad |f^{m_n+p_{n-1}}(\zeta)| = |f^{m_n}(f^{p_{n-1}}(\zeta))| \leq \frac{1}{16}M^{m_n}(|f^{p_{n-1}}(\zeta)|, f).$$

This is in contradiction to (3.9). We deduce that $\zeta \in A''(f) \cap J(f)$, as required. \square

4. PROOF OF THEOREM 3

We require the following well-known distortion lemma; see, for example, [3, Lemma 7].

Lemma 4.1. *Suppose that f is a transcendental entire function and that $U \subset I(f)$ is a simply connected Fatou component of f . Suppose that K is a compact subset of U . Then there exist $C > 1$ and $N \in \mathbb{N}$ such that*

$$\frac{1}{C}|f^n(z)| \leq |f^n(w)| \leq C|f^n(z)|, \quad \text{for } w, z \in K, n \geq N.$$

We also require the following [20, Lemma 10]. Here a set S is *backwards invariant* if $z \in S$ implies that $f^{-1}(\{z\}) \subset S$.

Lemma 4.2. *Suppose that f is a transcendental entire function, and that $E \subset \mathbb{C}$ is non-empty.*

- (a) *If E is backwards invariant, contains at least three points and $\text{int } E \cap J(f) = \emptyset$, then $J(f) \subset \partial E$.*
- (b) *If every component of $F(f)$ that meets E is contained in E , then $\partial E \subset J(f)$.*

We now prove Theorem 3, and so we suppose that f is a transcendental entire function. For part (a) of the theorem, let U be a simply connected Fatou component of f which meets $A(f)$. Note [21, Theorem 1.2] that $U \subset A(f)$. Suppose, by way of contradiction, that $z_0 \in U \cap A'(f)$. Then there exists $N \in \mathbb{N}$, a point $z_N = f^N(z_0)$ and $R = |z_N|$ such that

$$|f^n(z_N)| = M^n(R, f), \quad \text{for } n \in \mathbb{N}.$$

Let U_N be the Fatou component of f containing z_N , and note (see, for example, [22, Lemma 4.2]) that U_N is also simply connected. Choose a point $z'_N \in U_N$ such that $|z'_N| = R' < R$. Clearly

$$|f^n(z'_N)| \leq M^n(R', f), \quad \text{for } n \in \mathbb{N}.$$

Since there exists $N' \in \mathbb{N}$ such that $f^n(z'_N) \neq 0$, for $n \geq N'$, we deduce that

$$(4.1) \quad \frac{|f^n(z_N)|}{|f^n(z'_N)|} \geq \frac{M^n(R, f)}{M^n(R', f)}, \quad \text{for } n \geq N'.$$

Let $K = \{z_N, z'_N\}$. We deduce by Lemma 4.1 that the left-hand side of (4.1) is bounded for $n \geq N'$. However, it follows from Corollary 2.1 that the right-hand side of (4.1) tends to infinity as $n \rightarrow \infty$. This contradiction completes the proof of part (a) of the theorem.

Suppose next that U is a multiply connected Fatou component of f . Part (b) of the theorem is an immediate consequence of Lemma 3.3 and Theorem 1.

For part (c) of the theorem, we note that it follows from Theorem 2 that $A''(f) \cap J(f)$ is an infinite set, since for each $z \in A''(f) \cap J(f)$ at least one of the points $z, f(z)$ or $f^2(z)$ must have infinitely many preimages. It is known [3, Theorem 4] that the set of repelling periodic points of f is dense in $J(f)$. Clearly $A''(f)$ contains no periodic points, and so $\text{int } A''(f) \subset F(f)$. We deduce by Lemma 4.2(a), applied with $E = A''(f) \cap J(f)$ and then with $E = A''(f)$, that $A''(f)$ is dense in $J(f)$, and that $J(f) \subset \partial A''(f)$.

For part (d) of the theorem, suppose that $A'(f) \cap J(f) \neq \emptyset$. For the same reasons as above, it follows that $A'(f) \cap J(f)$ is an infinite set and $\text{int } A'(f) \subset F(f)$. We

deduce by Lemma 4.2(a), applied with $E = A'(f) \cap J(f)$ and then with $E = A'(f)$, that $A'(f)$ is dense in $J(f)$, and that $J(f) \subset \partial A'(f)$.

For part (e) of the theorem, suppose that $A'(f) \cap F(f) = \emptyset$. We deduce from part (a), Lemma 3.3 and Lemma 4.2(b) that $\partial A''(f) \subset J(f)$, and hence by part (c) that $J(f) = \partial A''(f)$.

Finally, for part (f) of the theorem, suppose that $A'(f) \cap J(f) \neq \emptyset$ and that $A'(f) \cap F(f) = \emptyset$. We deduce from Lemma 4.2(b) that $\partial A'(f) \subset J(f)$, and hence by part (d) that $J(f) = \partial A'(f)$.

5. EXAMPLES

Example 1. Let $f_1(z) = \exp(z)$. We observe that $\mathcal{M}(f_1) = [0, \infty)$ and that $f_1(\mathcal{M}(f_1)) \subset \mathcal{M}(f_1)$. We deduce that $A'(f_1)$ is the countable union of analytic curves

$$A'(f) = \bigcup_{k=0}^{\infty} f_1^{-k}([0, \infty)).$$

Since [3, Lemma 4] the set $\bigcup_{k=0}^{\infty} f_1^{-k}(-1)$ is dense in $J(f_1)$ and, as is well-known, $I(f_1) \subset J(f_1)$, we deduce that $\bigcup_{k=0}^{\infty} f_1^{-k}(-1)$ is dense in $A'(f_1)$. Thus

$$\bigcup_{k=0}^{\infty} f_1^{-k}((-\infty, 0)) \subset A'(f_1) \subset \overline{\bigcup_{k=0}^{\infty} f_1^{-k}((-\infty, 0))}.$$

Rempe [14, Proposition 3.1] showed that the set $\bigcup_{k=0}^{\infty} f_1^{-k}((-\infty, 0))$ is connected. We deduce that $A'(f_1)$ is connected.

Example 2. Let $f_2(z) = i \exp(z)$. We observe that $\mathcal{M}(f_2) = [0, \infty)$ and that $f_2(\mathcal{M}(f_2)) \cap \mathcal{M}(f_2) = \emptyset$. It follows that $A'(f_2) = \emptyset$.

Example 3. Baker [1] showed that constants $C > 0$ and $0 < a_1 < a_2 < \dots$ can be chosen in such a way that

$$f_3(z) = Cz^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right)$$

is a transcendental entire function with a multiply connected Fatou component. Since f_3 only has positive coefficients in its power series, it is clear that $A'(f_3)$ contains an unbounded interval in the positive real axis. We deduce by Lemma 3.2 that $A'(f_3) \cap F(f_3) \neq \emptyset$. It follows, by Theorem 3 part (b), that $\partial A''(f_3)$ and $\partial A'(f_3)$ both contain points of $F(f_3)$.

Example 4. Let $f_4(z) = i f_3(z)$, where f_3 is the function in Example 3. It can be seen from the construction in [1] that f_4 also has a multiply connected Fatou component. However, for the same reason as the function f_2 in Example 2, we see that $A'(f_4) = \emptyset$.

Example 5. We define a family of transcendental entire functions by

$$g_{\alpha}(z) = \alpha \exp(e^{z^2} + \sin z), \quad \text{for } \alpha > 0.$$

The function g_1 was considered by Hardy [8]. It is clear that Theorem 4 follows from the following lemmas.

Lemma 5.1. *If $\alpha > 0$, then $A'(g_{\alpha})$ is uncountable and totally disconnected.*

Lemma 5.2. *If $\alpha > 0$ is sufficiently small, then there are uncountably many singleton components of $A''(g_\alpha)$. Moreover, $A''(g_\alpha)$ has at least one unbounded component.*

Proof of Lemma 5.1. To show that $A'(g_\alpha)$ is uncountable and totally disconnected we show that $A'(g_\alpha)$ consists of the preimages of a set $S \subset \mathbb{R}$, and we use properties of the maximum modulus of g_α to show that S is uncountable and totally disconnected.

Although Hardy [8] considered only g_1 , it follows easily from his result that there exists $r_0 > 0$ such that the following holds. If $|z| \geq r_0$, then $z \in \mathcal{M}(g_\alpha)$ if and only if z is real, and lies on the positive real axis if $\sin |z| > 0$ and on the negative real axis if $\sin |z| < 0$. Since g_α maps points on the real axis to the positive real axis, we deduce that

$$(5.1) \quad A'(g_\alpha) = \bigcup_{k=0}^{\infty} g_\alpha^{-k}(S) \quad \text{where } S = \{x \geq r_0 : \sin g_\alpha^n(x) \geq 0, \text{ for } n \geq 0\}.$$

We prove first that S is uncountable. Let $I_k = [2k\pi, (2k+1)\pi]$, for $k \in \mathbb{N}$. We observe that $g'_\alpha(x) \rightarrow \infty$ as $x \rightarrow \infty$. Hence there exists $k_0 \geq r_0/2\pi$, $k_0 \in \mathbb{N}$, such that if $k \in \mathbb{N}$ and $k \geq k_0$, then there exists $k' > k$ such that $I_{k'} \cup I_{k'+1} \subset g_\alpha(I_k)$.

We construct an increasing sequence of integers $(k_n)_{n \in \mathbb{N}}$ as follows, noting that at each stage in the construction we have two distinct choices. First set either $k_1 = k_0$ or $k_1 = k_0 + 1$. Suppose inductively that k_n is defined for some $n \in \mathbb{N}$. There exists $k'_n > k_n$ such that $I_{k'_n} \cup I_{k'_n+1} \subset g_\alpha(I_{k_n})$. We choose either $k_{n+1} = k'_n$ or $k_{n+1} = k'_n + 1$. This completes the construction of the sequence. We then find a point $\zeta \in S$ by Lemma 3.1, with $E_n = I_{k_n}$ and $m_n = 1$, for $n \in \mathbb{N}$.

Now let S' be the subset of S consisting of points which can be constructed in this way. Since at each stage in the construction we had two distinct choices, there is a surjection from S' to the set of infinite binary strings. Hence S' , and so also S , is uncountable.

Next we prove that S is totally disconnected. Let $J_k = [(2k+5/4)\pi, (2k+7/4)\pi]$, for $k \in \mathbb{N}$. It follows from a construction similar to the above that if $k \geq r_0/2\pi$ is sufficiently large, then J_k contains uncountably many points in $\mathbb{R} \setminus S$ and indeed in $\mathbb{R} \setminus A'(g_\alpha)$. (We simply replace all references to the intervals I_n in the construction above to the intervals J_n , and thereby construct uncountably many points whose orbit lies in $\bigcup_{n \in \mathbb{N}} J_n$, in which case each point lies in $\mathbb{R} \setminus A'(g_\alpha)$.)

Suppose next that $x_1 < x_2$ are points in S . Since g_α has a large derivative on the real axis, we deduce that there exists an arbitrarily large k , and $N \in \mathbb{N}$ such that $g_\alpha^N(x_1) < (2k+5/4)\pi$ and $(2k+7/4)\pi < g_\alpha^N(x_2)$. Hence, there is a point $x' \in \mathbb{R} \setminus S$ such that $g_\alpha^N(x_1) < x' < g_\alpha^N(x_2)$. Since g_α is increasing on \mathbb{R} , there is a point x'' such that $x_1 < x'' < x_2$ and $g_\alpha^N(x'') = x'$. It follows that $x'' \notin S$, and we deduce that S is totally disconnected, as required.

It is known ([13, Lemma 2.5] and see also [11, Chapter II]) that a countable union of compact, totally disconnected subsets of \mathbb{C} is totally disconnected. We deduce from this result, from the fact that S is a countable union of compact, totally disconnected sets, and from (5.1) that $A'(g_\alpha)$ is indeed totally disconnected. \square

Proof of Lemma 5.2. Consider first the real-valued function $x \rightarrow g_\alpha(x)$. It is straightforward to show that $g''_\alpha(x) > 0$, for $x > 0$. We may assume, therefore,

that $\alpha > 0$ is sufficiently small that $g_\alpha(x)$ has exactly two fixed points. An elementary calculation shows that there is an attracting fixed point $p_\alpha \in (0, 1)$, and a repelling fixed point $q_\alpha \in (p_\alpha, \infty)$.

Suppose that $x \in (q_\alpha, \infty)$. Then there exists $N = N(x) \in \mathbb{N}$ such that, by the result of Hardy [8] mentioned in the proof of Lemma 5.1,

$$(5.2) \quad g_\alpha^{n+1}(x) \geq \frac{1}{e^2} \max\{g_\alpha(g_\alpha^n(x)), g_\alpha(-g_\alpha^n(x))\} = \frac{1}{e^2} M(g_\alpha^n(x), g_\alpha), \quad \text{for } n \geq N.$$

It follows by [21, Theorem 2.9] that $x \in A(g_\alpha)$. Hence $(q_\alpha, \infty) \subset A(g_\alpha)$.

It was shown in the proof of Lemma 5.1 that, for sufficiently large values of $k \in \mathbb{N}$, $[(2k+5/4)\pi, (2k+7/4)\pi] \setminus A'(g_\alpha)$ is uncountable. We deduce that $(q_\alpha, \infty) \cap A''(g_\alpha)$ is uncountable. We claim that $(q_\alpha, \infty) \cap A''(g_\alpha)$ is totally disconnected. Suppose that $x_1 < x_2$ are points in $(q_\alpha, \infty) \cap A''(g_\alpha)$. Arguing as in the proof of Lemma 5.1, we deduce that there exists an arbitrarily large k , and $N \in \mathbb{N}$ such that $g_\alpha^N(x_1) < 2k\pi$ and $(2k+1)\pi < g_\alpha^N(x_2)$. It follows that there is a point $x' \in A'(g_\alpha)$ such that $g_\alpha^N(x_1) < x' < g_\alpha^N(x_2)$. Since g_α is increasing on \mathbb{R} , there is a point x'' such that $x_1 < x'' < x_2$ and $g_\alpha^N(x'') = x'$. It follows that $x'' \notin A''(g_\alpha)$, and we deduce that $(q_\alpha, \infty) \cap A''(g_\alpha)$ is totally disconnected, as claimed.

We now show that there exists $\alpha_0 > 0$ such that

$$(5.3) \quad (q_\alpha, \infty) \text{ is a component of } A(g_\alpha), \quad \text{for } 0 < \alpha < \alpha_0.$$

Since $(q_\alpha, \infty) \cap A''(g_\alpha)$ is uncountable and totally disconnected, the fact that $A''(g_\alpha)$ has uncountably many singleton components, for $0 < \alpha < \alpha_0$, then follows from (5.3).

The proof of (5.3) is complicated. Roughly speaking, our method is as follows. First we choose α sufficiently small that $F(g_\alpha)$ has an unbounded attracting basin U , which lies outside the escaping set and contains infinitely many preimages of the positive imaginary axis. We then deduce (5.3) by a careful analysis of the properties of these preimages, and from the properties of the fast escaping set.

First we note some facts about the function g_α . The proof is simplified slightly by noting that g_α satisfies the equation

$$(5.4) \quad g_\alpha(\bar{z}) = \overline{g_\alpha(z)}, \quad \text{for } z \in \mathbb{C}.$$

We observe that if $z = x + iy$ and $g_\alpha(z) = Re^{i\theta} = u + iv$, then, by a calculation,

$$(5.5) \quad \log R = e^{x^2-y^2} \cos 2xy + \sin x \cosh y + \log \alpha,$$

$$(5.6) \quad \theta = e^{x^2-y^2} \sin 2xy + \cos x \sinh y,$$

$$(5.7) \quad \frac{\partial u}{\partial x} = u(e^{x^2-y^2}(2x \cos 2xy - 2y \sin 2xy) + \cos x \cosh y - \tan \theta(e^{x^2-y^2}(2x \sin 2xy + 2y \cos 2xy) + \sin x \sinh y)),$$

$$(5.8) \quad \frac{\partial v}{\partial y} = v(e^{x^2-y^2}(-2y \cos 2xy - 2x \sin 2xy) + \sin x \sinh y + \cot \theta(e^{x^2-y^2}(-2y \sin 2xy + 2x \cos 2xy) + \cos x \cosh y)).$$

Let $\Gamma_0 = \{z : \arg z = \pi/2\}$ and $\gamma_0 = \{z : \arg z = \pi/4\}$. We deduce from (5.5) that g_α is bounded on Γ_0 .

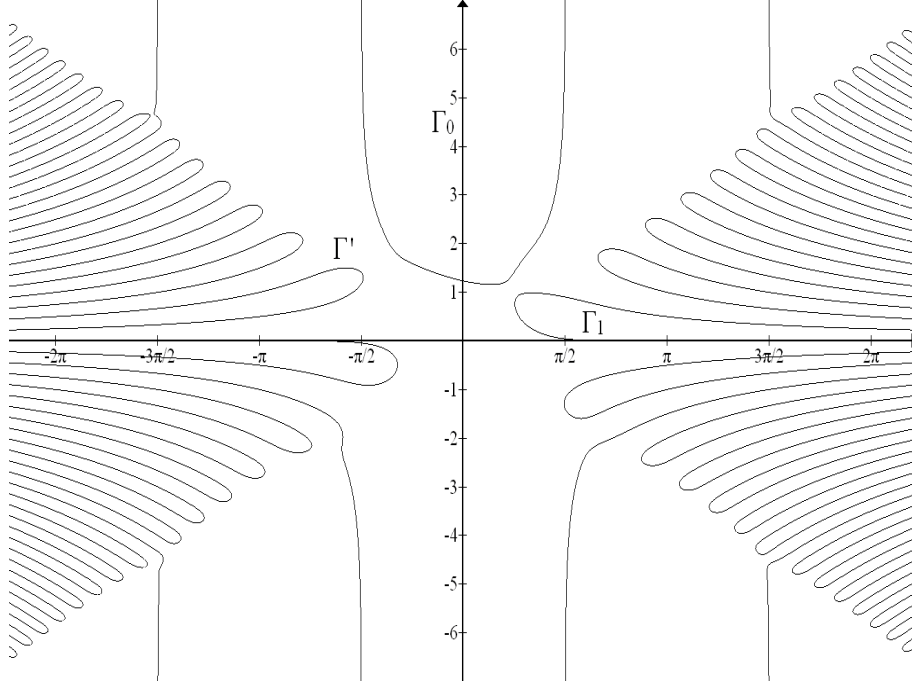


FIGURE 1. Some of the preimages of the positive imaginary axis, Γ_0 , calculated by solving equation (5.6) for $\theta = \pi/2$.

Next we define three domains $V'' \subset V' \subset V$ in which g_α has certain useful properties. First, for a large value of $K > 1$, which we fix later, let V be the domain

$$V = \{z = x + iy : x > K, 0 < y < e^{-x^2}\}.$$

It follows from (5.5) and (5.6) that we can choose K sufficiently large that, for $z = x + iy \in V$,

$$(5.9) \quad e^{x^2} - 2 \leq \log \frac{|g_\alpha(z)|}{\alpha} \leq e^{x^2} + 2 \quad \text{and} \quad 2y(xe^{x^2} - 1) \leq \arg g_\alpha(z) \leq 2y(xe^{x^2} + 1).$$

We deduce that V has unbounded intersection with a component of $g_\alpha^{-1}(\Gamma_0)$. Let Γ_1 be the intersection of this preimage component with V , and note by (5.9) that

$$y \sim \frac{\pi}{4x} e^{-x^2} \text{ as } x \rightarrow \infty, \quad \text{for } x + iy \in \Gamma_1.$$

It follows by differentiating (5.6) that, increasing the size of K if necessary, we may assume that

$$(5.10) \quad \frac{dy}{dx} < 0, \quad \text{for } x + iy \in \Gamma_1.$$

We deduce also that V has unbounded intersection with a component of $g_\alpha^{-1}(\gamma_0)$. Let γ_1 be the intersection of this preimage component with V , and note by (5.9) that

$$y \sim \frac{\pi}{8x} e^{-x^2} \text{ as } x \rightarrow \infty, \quad \text{for } x + iy \in \gamma_1.$$

Increasing the size of K if necessary, we may assume that Γ_1 and γ_1 each intersect the boundary of V only at a point with real part K and modulus less than $2K$. We may also assume that if $K < x < x'$, then

$$(5.11) \quad \text{if } x + iy \in \gamma_1, \text{ then there exists } x' + iy' \in \Gamma_1 \text{ such that } y' < 4y.$$

Let $V' \subset V$ be the domain bounded by Γ_1 , $\{z : \operatorname{Re}(z) = K\}$, and the real axis. Increasing the size of K again, if necessary, we may assume, by (5.9), that

$$(5.12) \quad 0 < \arg g_\alpha(z) < 3\pi/4, \quad \text{for } z \in V'.$$

Then let $V'' \subset V'$ be the domain bounded by γ_1 , $\{z : \operatorname{Re}(z) = K\}$, and the real axis. We deduce from (5.7) and (5.8) that if $x + iy \in V'' \cup \gamma_1$ and $g_\alpha(x + iy) = u + iv$, then as $x \rightarrow \infty$,

$$\frac{\partial u}{\partial x} = u \left(2xe^{x^2} + O(x) \right) \quad \text{and} \quad \frac{\partial v}{\partial y} = v \left(2xe^{x^2} \cot(\arg g_\alpha(z)) + O(x) \right).$$

Hence, by (5.9), may assume that

$$(5.13) \quad \frac{\partial}{\partial x} \operatorname{Re}(g_\alpha(x + iy)) > 0 \quad \text{and} \quad \frac{\partial}{\partial y} \operatorname{Im}(g_\alpha(x + iy)) > 0, \quad \text{for } x + iy \in V'' \cup \gamma_1.$$

Finally, by a similar argument to the one given earlier relating to Γ_1 , we note that there is a component Γ' of $g_\alpha^{-1}(\Gamma_0)$ which is asymptotic to the negative real axis. We omit the details. Increasing K one last time, if necessary, we may assume that $B(0, K) \cap \Gamma' \neq \emptyset$. Note that Γ_1 , γ_1 and Γ' are each independent of α .

Next we fix a value of α_0 . We choose $0 < \alpha_0 < e^{-1}$ sufficiently small that if $0 < \alpha < \alpha_0$, then g_α has the two fixed points p_α and q_α discussed earlier, and also that

$$g_\alpha(B(0, 2K) \cup \Gamma_0) \subset B(0, 1).$$

Suppose then that $0 < \alpha < \alpha_0$. We deduce that g_α has an unbounded simply connected Fatou component, U , which contains Γ_0 , the disc $B(0, 2K)$ and so also Γ_1 and Γ' , the attracting fixed point p_α , and indeed the interval $(0, q_\alpha)$. Note that $q_\alpha \geq 2K$, but we do not assume that equality holds. Clearly $U \cap A(g_\alpha) = \emptyset$, and so all preimages of Γ_0 lie outside $A(g_\alpha)$.

Clearly $q_\alpha \in J(g_\alpha) \cap \mathbb{C} \setminus A(g_\alpha)$. We claim that $(q_\alpha, \infty) \subset J(g_\alpha)$. Since g_α is bounded on Γ_0 , it follows by Lemma 3.2 that g_α has no multiply connected Fatou component. Moreover, the set $A'(g_\alpha)$ is dense in (q_α, ∞) . We deduce by Theorem 3 part (a) that $(q_\alpha, \infty) \subset J(g_\alpha)$, as claimed.

Suppose next that $U' \neq U$ is a component of $F(g_\alpha)$ such that $g_\alpha(U') \subset U$. We claim that $U' \cap V' = \emptyset$. Suppose, to the contrary, that $U' \cap V' \neq \emptyset$. Since the boundary of V' consists of points either in U or in $J(g_\alpha)$, we deduce that $U' \subset V'$. Now [10, Theorem 3] $g_\alpha(U')$ and U may differ by at most two points. However, $\Gamma' \subset U$ contains points with argument arbitrarily close to π . This is a contradiction, by (5.12), completing the proof of our claim.

For $\xi > K$, let $y(\xi)$ be such that $\xi + iy(\xi)$ is the point on γ_1 of real part ξ ; this point is unique by (5.13). By (5.9), we can choose $\xi > q_\alpha$ sufficiently large that

$$(5.14) \quad \operatorname{Re}(g_\alpha(\xi + iy')) > 2\xi \quad \text{and} \quad \operatorname{Im}(g_\alpha(\xi + iy')) > 8y', \quad \text{for } y' \in (0, y(\xi)].$$

We claim that if $y' > 0$, then there is a curve $\gamma' \subset U$ such that

$$(\gamma' \cap \{z : \operatorname{Re}(z) \geq \xi\}) \subset \{z : 0 < \operatorname{Im}(z) < y'\}.$$

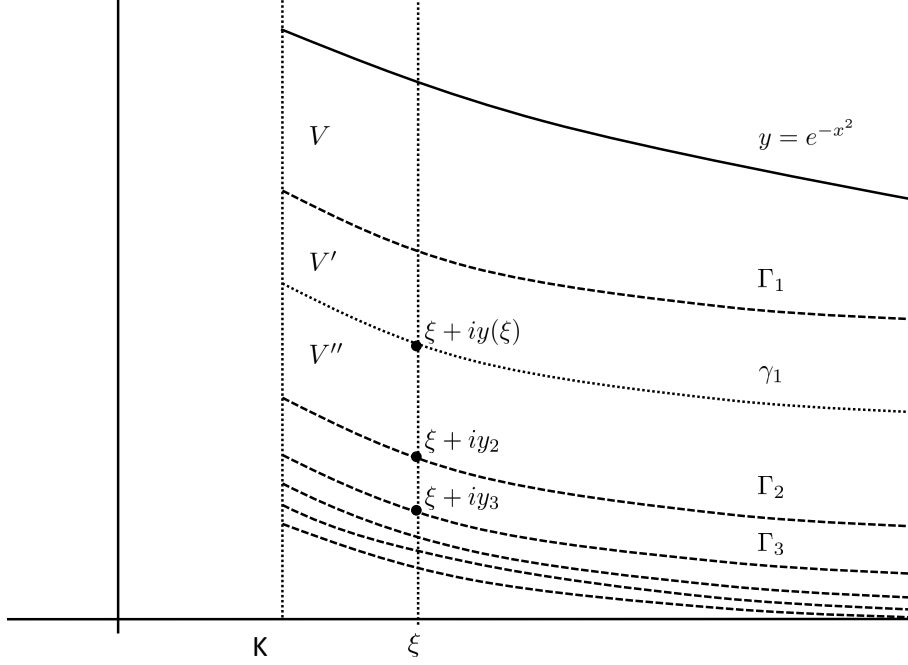


FIGURE 2. Some of the preimages of Γ_0 , shown dashed, constructed in this proof.

We construct sequences of disjoint curves $(\Gamma_n)_{n \geq 2}$ and positive real numbers $(y_n)_{n \geq 2}$ such that $\Gamma_n \subset U \cap V''$ is an unbounded curve with a finite endpoint on the line $\{z : \operatorname{Re}(z) = K\}$, for $n \geq 2$; see Figure 2.

We first define Γ_2 and y_2 . It follows by (5.11), (5.13) and (5.14) that there exists $0 < y_2 < y(\xi)$ such that $g_\alpha(\xi + iy_2) \in \Gamma_1 \subset U$. Let Γ_2 be the intersection of V'' with the component of $g_\alpha^{-1}(\Gamma_1)$ containing $\xi + iy_2$. By the comments above, Γ_2 cannot lie in a preimage component of U distinct from U , and so must lie in U . Also Γ_2 is unbounded and has a finite endpoint on the line $\{z : \operatorname{Re}(z) = K\}$ by (5.13) and since it cannot intersect with γ_1 or with the real axis.

Suppose next that, inductively, we have defined Γ_k and y_k , for some $k \geq 2$. By (5.13) and (5.14), there exist $0 < y_{k+1} < y_k$ such that $g_\alpha(\xi + iy_{k+1}) \in \Gamma_k \subset U$. Let Γ_{k+1} be the intersection of V'' with the component of $g_\alpha^{-1}(\Gamma_k)$ containing $\xi + iy_{k+1}$. We have that $\Gamma_{k+1} \subset U$ for the same reasons that applied to Γ_2 . Finally Γ_{k+1} is unbounded and has a finite endpoint on the line $\{z : \operatorname{Re}(z) = K\}$ by (5.13) and since it cannot intersect with Γ_k or with the real axis. This completes the construction of the sequences.

Suppose that $n \geq 2$. It follows by (5.10) and (5.13) that

$$y_n = \max\{\operatorname{Im}(z) : z \in \Gamma_n \text{ and } \operatorname{Re}(z) \geq \xi\}.$$

It also follows by (5.13) and (5.14) that $y_{n+1} < y_n/8$, for $n \in \mathbb{N}$. This completes the proof of the claim following equation (5.14).

Let W be the component of $\mathbb{C} \setminus U$ containing $[q_\alpha, \infty)$, and let $W' = W \setminus [\xi, \infty)$. We claim that W' is bounded. To prove this we let Γ'_1 be the curve in U formed

by the union of Γ_1 , the complex conjugate of Γ_1 (recall (5.4)), and the part of the line $\{z : \operatorname{Re}(z) = K\}$ joining the finite endpoints of these curves. Let S be the bounded component of $\mathbb{C} \setminus (\Gamma'_1 \cup \{z : \operatorname{Re}(z) = \xi\})$. We claim that, in fact, $W' \subset S$. For, suppose that $z \in W' \cap (\mathbb{C} \setminus S)$. If z lies in the component of $\mathbb{C} \setminus \Gamma'_1$ which does not contain $[q_\alpha, \infty)$, then $\Gamma'_1 \subset U$ separates z from $[q_\alpha, \infty)$. On the other hand, if z lies in the component of $\mathbb{C} \setminus \Gamma'_1$ which does contain $[q_\alpha, \infty)$, then $\operatorname{Re}(z) \geq \xi$ and $\operatorname{Im}(z) \neq 0$. Recalling the sequence of disjoint curves $(\Gamma_n)_{n \geq 2}$ constructed earlier, we see that z can be separated from $[q_\alpha, \infty)$ by the curve in U formed by the union of Γ_n , for some $n \in \mathbb{N}$, the complex conjugate of Γ_n , and the part of the line $\{z : \operatorname{Re}(z) = K\}$ joining the finite endpoints of these curves. Hence W' is bounded as claimed.

Let T be the component of $A(g_\alpha)$ containing (q_α, ∞) . Suppose, contrary to (5.3), that $T \neq (q_\alpha, \infty)$. Then, since $q_\alpha \notin A(g_\alpha)$, there exists $\zeta \in T$ such that $\operatorname{Im}(\zeta) \neq 0$. We have that $\zeta \in W'$ as otherwise U separates ζ from (q_α, ∞) .

It follows from (1.1) that there exist $\ell \in \mathbb{Z}$ and $R > 0$ such that $M^n(R, g_\alpha) \rightarrow \infty$ as $n \rightarrow \infty$ and $\zeta \in A_R^\ell(g_\alpha)$, where

$$(5.15) \quad A_R^\ell(g_\alpha) = \{z : |g_\alpha^n(z)| \geq M^{n+\ell}(R, g_\alpha), \text{ for } n \in \mathbb{N}, n + \ell \in \mathbb{N}\} \subset A(g_\alpha).$$

Let T' be the component of $A_R^\ell(g_\alpha)$ containing ζ . We have [21, Theorem 1.1] that T' is closed and unbounded. Since $T' \cap U = \emptyset$ and W' is bounded, T' contains $[\xi, \infty)$. We deduce also that $g_\alpha(T') \subset T'$.

Let $\xi' > q_\alpha$ be the smallest value such that $[\xi', \infty) \subset T'$. Since W' is bounded, by (5.15) there exists $N \in \mathbb{N}$ such that

$$g_\alpha^N(T') \cap W' = \emptyset.$$

Since T' is connected and $g_\alpha^N(T') \subset T'$, we deduce that $g_\alpha^N(T') \subset [\xi, \infty)$. Hence T' contains a curve T'' such that $\{\zeta\} \cup [\xi', \infty) \subset T''$.

Choose ξ'' such that $q_\alpha < \xi'' < \xi'$. Set $\widehat{T} = T'' \cup [\xi'', \xi')$. We note that there exists $N' \in \mathbb{N}$ such that $g_\alpha^{N'}(\widehat{T}) \subset [\xi, \infty)$. Hence there is no neighbourhood of ξ' in which $g_\alpha^{N'}$ is a homeomorphism. This is a contradiction, since $g'_\alpha(x) > 0$, for $x > 0$. This contradiction completes our proof of (5.3). Hence, as already observed, (q_α, ∞) contains uncountably many singleton components of $A''(g_\alpha)$.

Finally we show that $A''(g_\alpha)$ has at least one unbounded component. We recall that g_α has no multiply connected Fatou component. Hence we may suppose that $\zeta \in A''(g_\alpha)$ is the point constructed in the proof of Theorem 2, with $f = g_\alpha$. We note that since $|g_\alpha(z)| < 1$, for $z \in \Gamma_0$, we may assume that the sequence $(m_k)_{k \in \mathbb{N}}$, also constructed in the proof of Theorem 2, satisfies $m_k = 1$, for $k \in \mathbb{N}$.

Let Q be the component of $A''(g_\alpha)$ containing ζ , and suppose that Q is bounded. Let Q' be the component of $A(g_\alpha)$ containing ζ . Since [21, Theorem 1.1] all components of $A(g_\alpha)$ are unbounded, $Q' \neq Q$. In particular Q' contains a point $\zeta' \in A'(g_\alpha)$.

Since $\zeta' \in A'(g_\alpha)$, there exists $N \in \mathbb{N}$ such that $g_\alpha^N(\zeta') \in (q_\alpha, \infty)$. Since (q_α, ∞) is a component of $A(g_\alpha)$, we obtain that $g_\alpha^N(Q') \subset (q_\alpha, \infty)$. In particular we have that $g_\alpha^N(\zeta) \in (q_\alpha, \infty)$.

We deduce, by (5.2) with $x = g_\alpha^N(\zeta)$, that there exists $N' \in \mathbb{N}$ such that

$$(5.16) \quad g_\alpha(g_\alpha^{n'}(\zeta)) \geq \frac{1}{e^2} M(g_\alpha^{n'}(\zeta), g_\alpha), \quad \text{for } n \geq N'.$$

This is a contradiction to (3.11), and so Q is unbounded as required. \square

Example 6. We define a family of transcendental entire functions by

$$h_\alpha(z) = \alpha e^z, \quad \text{for } \alpha \in (0, e^{-1}).$$

This family is contained in the *exponential family*, the dynamics of which have been widely studied. It is well-known that h_α has an unbounded simply connected Fatou component, which contains the imaginary axis and an attracting fixed point, and also a repelling fixed point $q > 1$.

As in Example 1, we see that $(q, \infty) \subset A'(h_\alpha)$. The techniques of the proof of Lemma 5.2 may be used to show that (q, ∞) is a component of $A(h_\alpha)$. We omit the details.

Acknowledgment: The author is grateful to Gwyneth Stallard and Phil Rippon for all their help with this paper, and also John Osborne for his drawing his attention to [14].

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